A Physics-based Model for Estimating the Shadow Ground Speed

Although we can measure the speed of the lunar shadow on the ground and use this to create a function that predicts the speed from the observer's longitude (see Problem Set 11) from $V(L) = 1.06L^2 + 179\ L + 9900$, this is an empirical model that dies not explain why the coefficients are as shown. We merely 'fit' a quadratic function to the data points and then use this function to interpolate to other longitudes. What we would really like to do is to understand the motion of the moon, Earth and sun, and use this to create a new function that explains the form of the equation in relation to physical parameters like distance, speed and time. The common thing to do is to take a step-by-step approach and each time add an improvement to the previous understanding by considering additional physical factors.

Version 1.0. The shadow speed on the ground is related to the true speed of the shadow, which is just the speed of the moon in its orbit, reduced by the rotational speed of Earth at the location of the shadow. At the time of the eclipse, the orbital speed of the moon is 3,873 km/h. The rotation speed is just Vrot=1674cos(latitude).

But because the Earth is also in motion we have to allow for this. First, Earth moves around its barycenter:

At the time of the eclipse, the distance between the center of Earth and moon is 372,000 km. The mass of Earth and moon balance at the barycenter. The mass of the moon is 0.0123Mearth, so the distance from the center of Earth to the barycenter is just d = 372000 x (0.0123/1.0123) = 4520 km, and for the moon it is 372000-4520 = 367,480 km. Now these bodies orbit the barycenter counterclockwise and we can calculate their speeds for a lunar sidereal month of 27.3 days, so for Earth we get $V_b = 2\pi$ 4520km /(27.3 x 23.933) = 43 km/hr.

Next, Earth moves in its orbit around the sun during the time of the eclipse. This means that the position of the sun changes slightly and this changes the movement of the shadow. As seen from the sun, during the 1 hour that the shadow sweeps across the United States, the Earth-moon system defined by its barycenter moves in its orbit.

$$\text{Angle} = \frac{360 \text{ degrees}}{365 \text{ days} \times 23.93\text{h/d}} = 0.0412 \text{ degrees/hour.}$$

This slight angular shift, when projected to the surface of Earth located (372,000 km – 6378 km) = 365,600 km from the center of the moon corresponds to a distance (using the skinny-triangle approximation) of d = 0.0412/57.3 x 365,600 km = 263 kilometers each hour. So as seen from the surface of Earth, the solar 'light source' behind the moon shifts by 263 km/hr in a direction from east to west on the day-side of Earth, as Earth orbits the sun counterclockwise. Correcting for this motion we have to subtract 263 km/hr from the lunar shadow speed.

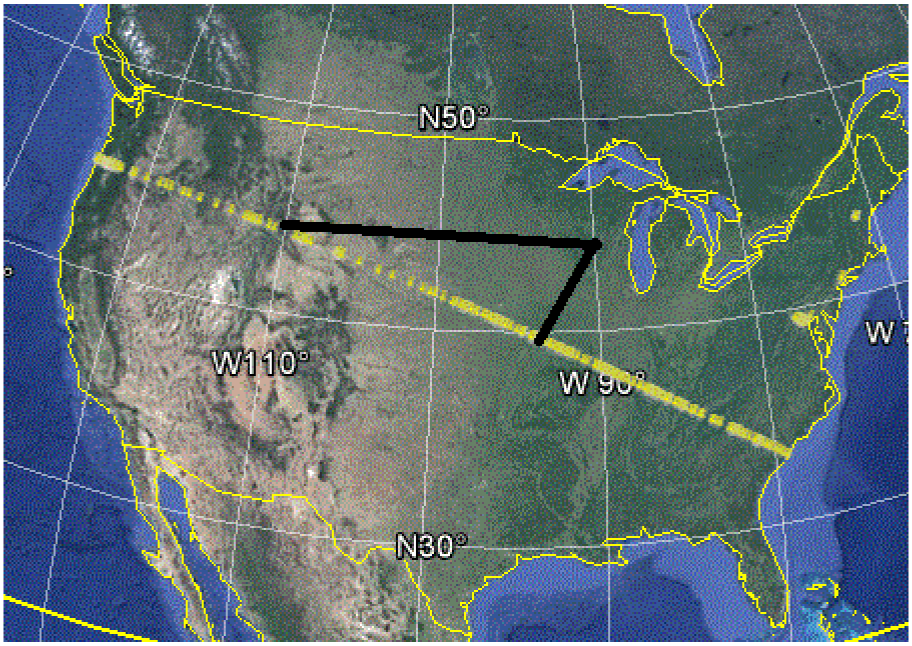
Shadow speed = 3,873 km/h –43 km/h – 263 km/h - 1674cos(latitude)

Vs(L) = 3567 – 1674cos(L)

Problem 1 – The measured shadow speeds at three locations along the path of totality are given in the table below. A) From our function Vs(L), what would we predict as the shadow speeds at these latitudes? B) What pattern do you see in the differences?

Location	Latitude (L)	Actual speed	Predicted	Difference
Madras, OR	44.7	3,750		
Casper, WY	42.8	2,800		
Carbondale, IL	37.6	2,350		
McClellanville, SC	33.0	2,400		

Version 2.0. Our first version was actually pretty successful over much of its range! But let's see if we can't improve it. One thing we notice is that the path of the eclipse is tilted at about 23^{o} to lines of longitude, so the speed of Earth's rotation along the line of totality is given by the geometry:

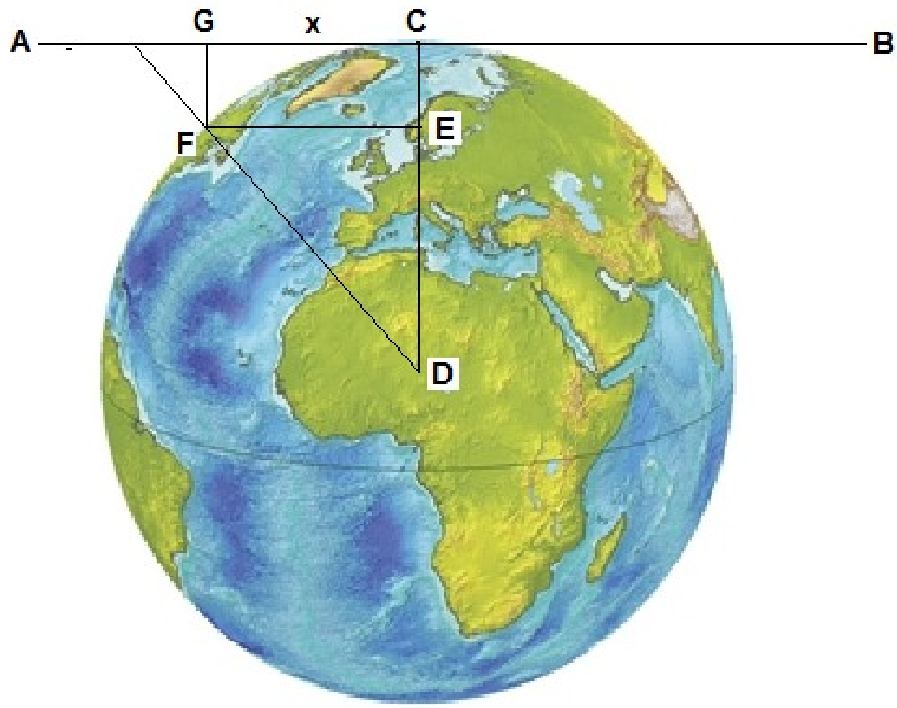


The West-East rotation speed V(rot) is drawn along the hypotenuse of the triangle, the component of this speed along the track is just

V(track)=V(rot)cos(23) so
Vs(L) = cos(23) x (3567 – 1674cos(L))

This reduces all of our predicted speeds by cos(23) = 0.92. Unfortunately, when we compute our predictions for the above table. We are underestimating the speeds even more than before by an additional 8%. Nevertheless, we still have to include this factor geometric factor in our models.

Version 3.0. How do we calculate the shadow speed in our data? We do this by timing how long it takes the moon's shadow to pass over our location. But we know from Problem 11 that the shadow diameter changes all along the path of totality from just under 100 km in Oregon to nearly 115 km in South Carolina. That's a 15% change! The problem is geometric distortion. Here is a sketch of what things look like.



The moon's shadow doesn't know anything about the curvature of the screen that it is being projected upon. All it knows is that the vertex of its shadow wants to travel through space parallel to the moon's orbit at that time, which is basically a straight line in space. The shadow is moving parallel to the tangent line AB at the time of the eclipse, and the dark spot is being projected onto the arc FC. Only close-by point C do we get an undistorted view of the lunar shadow, which looks like a circle. As we move to the left (west) or east (right) of this spot, the shadow turns into an ellipse.

On the tangent plane (AB) the length of the segment GC is defined by x measured from the origin at the tangent point C, and this is the same as the length of the segment FE.

The distance along the surface of Earth (DC=R=6378 km) represented by the arc FC is just $L = 2\pi R \times \theta/360$, where $\theta = m|FC|$ is the measure of the arc in degrees.

The segment FE has a length defined by $R\sin(\theta)$ where $\theta = m|FC|$.

From this we see that $x = R\sin(\theta)$ and $L=2\pi R\theta/360$ and so with a little algebra $\theta = 360L/2\pi R$ and by substitution, $x = R\sin(360L/2\pi R)$.

So, from a knowledge of how far you are, L, from the midpoint of the track of Totality, you can figure out where you are located in the tangent plane in terms of the linear variable x.

For example, the location of the path on the Oregon coast at First Contact is about 2000 km from the midpoint. The arc has an angular measure of 360*(2000/2πx6378) = 18 degrees. Then from x=Rsin(18) we get x = 1970 km.

The speed we measure on the ground is the surface speed of the shadow, but what we really have is information about the speed of the shadow in the tangent plane along AB in the figure. The relationship between these is just V(tangent) = cos(θ) V(surface).

At the tangent point, θ=0 in South Carolina, and V(tangent) = V(surface) but at the initial point on the Oregon coast where $\theta = 36^{o}$, we have V(tangent) = cos(36)V(surface) so the tangent speed is only 81% of the surface speed. Alternatively, the surface speed is 1.24 times faster than the tangent speed.

So, our previous model Vs = cos(23) [3567 – 1674cos(L)] is actually describing the tangent speed. To convert it into the surface speed we divide by cos(θ) where θ is the angle along the path of totality from McClellenville (θ=0) to our observation spot.

Problem 2 - Use the new model for the surface speed to calculate the predicted speeds and their differences with the actual speeds. What is the range of percentage differences relative to the actual speeds?

$$Vs = \cos(23)\,[\,3567 - 1674\cos(L)\,] / \cos(\theta)$$

Location	Latitude	θ	Obs. Speed	Predicted	Difference
Madras, OR	44.7	34^{o}	3,750		
Casper, WY	42.8	23^{o}	2,800		
Carbondale, IL	37.6	9^{o}	2,350		
McClellanville, SC	33.0	0^{o}	2,400		

Version 4.0. Our model would work if the lunar shadow were perpendicular to the surface of Earth, but in fact the eclipses at the various locations are viewing the sun and moon at different elevations above the horizon. This causes a second distortion to the shape of the lunar shadow. From geometry, the diameter of the shadow will be its circular diameter divided by sin(elevation). As the sun and moon are lower towards the horizon, the shadow becomes elongated.

$$Vs = \cos(23)\,[\,3567 - 1674\cos(L)\,] / [\cos(\theta)\,\sin(elev)]$$

Problem 3 – Recalculate the predictions with this new factor included. How do the percentage changes vary along the eclipse track?

Location	Latitude	θ	Elev	Obs	Predicted	Difference
Madras, OR	44.7	34^o	42^o	3,750		
Casper, WY	42.8	23^o	54^o	2,800		
Carbondale, IL	37.6	9^o	64^o	2,350		
McClellanville, SC	33.0	0^o	61^o	2,400		

Version 5.0 - In Version 3.0 we corrected for the difference between the tangent speed and the surface speed, but we still have to look at the change in the lunar shadow diameter. Recall from the above figure that along the tangent plane:

$Y = R - R\cos(\theta)$

The difference between Oregon and SC is 36 degrees, so compared to SC, the shadow is located

Y = 6378(1-cos(36)) = 1218 km below the tangent plane at SC.
X = 2x1970 = 3940 km along tangent plane from SC

The radius of the moon shadow on Earth's surface tangent plane over North America is found from the proportion:

$$\frac{1737}{377{,}700} = \frac{H}{(5673+6378-1218)}$$

$$H = \frac{(5673+6378-1218)\ 1737}{377{,}700} = 49.8 \text{ km radius or 99 km diameter}$$

The predicted and actual shadow diameters match almost exactly: 99 vs 99.7 km in Oregon and 115 vs 110 in SC.

Now the eclipse midpoint time is the time needed for the shadow to travel its own diameter along the surface. That depends on the speed of the shadow and its diameter. So the midpoint times include information about the shadow width along the track.

So….we correct the speeds for a function that depends on the shadow diameter.

Create geometric function $x(\theta) = 57.5/H(\theta)$

Where $H(\theta) = \dfrac{(5673+6378-6378(1-\cos(\theta)))\ 1737}{377{,}700}$

$H(\theta) = 0.0046\ (12051 - 6378(1-\cos(\theta)))$

Actual = (differential ground speed from midpoints) x $X(\theta)$
Where $X(\theta) = 57.5/(0.0046(12051-6378(1-\cos(\theta)))$

The dependence of this differential speed measurement on the lunar distance is as follows:

Speeds:
Earth rotation = 1674cos(latitude) km/h
Moon = 3873 km/h
Barycenter = 43 km/h
Solar = 273 km/h

Where: $X(\theta) = 57.5/(0.0046(12051-6378(1-\cos(\theta)))$ (Note: 0.0046 = 1737/377600)

S = cos(23)(Moon speed – Earth rot – barycenter – solar) X(D)

$$X(D) = \frac{115/2\ [\ 377{,}600\ ]}{1737[\ \ (377{,}600 - D) + 6378 - 6378(1-\cos(\theta)\ \ ]}$$

RX = d ((151,390,000-D) + X)

X = d(151390000-D)/(R-d) = shadow cone length = 377,600 km

So:

$$X(D) = \frac{115/2\ [\ d(151390000-D)/(R-d)\ ]}{1737(\ \ [d/(R-d)][(151390000-D) - D + 6378\cos(\theta)\ \ ]}$$

θ = angle along path of totality
R = 696,300 km and
d = 1737 km

$$X(D) = \frac{8.27\text{x}10^{-5}(151390000\text{-}D)}{378475 - 1.0025D + 6378\cos(\theta)}$$

Example: for D = 372,000 $\theta = 34.1$

X = 12489 / [5455+ 5287] = 1.14

So:

$$\text{Speed} = \cos(23) \times 8.27\text{x}10^{-5} \frac{(\text{Moon speed} - \text{Earth rot} - \text{barycenter} - \text{solar})\,(151390000\text{-}D)}{378475 - 1.0025D + 6378\cos\theta}$$

And for the speeds:
Earth = 1674cos(latitude)
Moon = 3873
Barycenter = 43
Solar = 273

We have:

$$\text{Speed} = \cos(23)6.61\text{ x}10^{-5} \frac{(3557 - 1674\cos(\text{Lat}))\,(151390000)}{378475 - 1.0025D + 6378\cos\theta}$$

Where we approximate 151390000-D by just 151390000 because
D <<< 151 million km!.

For D=372,000 km, θ=34 and lat=44.7,

Speed = 9211 (2367)/(5545+5287) = 2014 km/hr

Speed = 2181 km/hr. Now multiply by 1/cos(θ) to get ground speed.

2181 / cos(34) = 2630 km/s

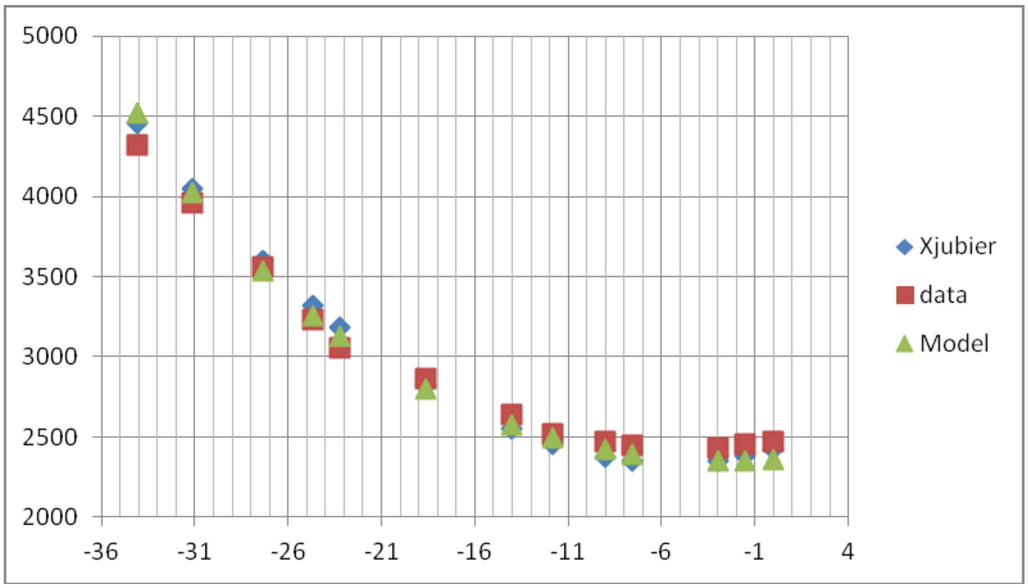
Now correct for the solar elevation angle 1/sin(41.6)

2630 /sin(41.6) = 3961 km/hr this is lower than actual value of 4460 km/hr by 11%

So:

$$\text{Speed} = \cos(23)6.61\times10^{-5}\,\frac{(3557 - 1674\cos(\text{Lat}))\,(151390000)}{(378475 - 1.0025D + 6378\cos\theta)\cos(\theta)\sin(\text{elev})}$$

Modeled speed vs actual instantaneous ground speed match each other to better than 11% accuracy. Note the model is consistently 81% of the actual value. If we add this mysterious fudge factor of 1.23, our predicted speeds would all match actual speeds to within 1%.



Our final function for the ground speed of the lunar shadow now looks like this:

$$\mathbf{Speed} = \mathbf{1.23\cos(23)6.61\times10^{-5}}\,\frac{\mathbf{(3557 - 1674\cos(Lat))\,(151390000)}}{\mathbf{(378475 - 1.0025D + 6378\cos\theta)\cos(\theta)\sin(elev)}}$$

Where:

23^{o} is the tilt of the path of totality to local meridian.

Lat = The latitude of the observer

θ = The angular distance between South Carolina and the observation point along the path of totality.

Elev = elevation angle of the sun above the horizon at Totality.

D = distance to the moon in km.

151390000 = distance from center of sun to center of Earth in km

1674 = equatorial rotation speed of Earth in km/h

3557 = Moon speed – solar – barycenter in km/h

378475 = comes from R = 696,300 km solar radius, d = 1737 km lunar radius,
151390000 km = earth-sun distance, and $d/(R-d) = 0.0025$
So $(d/R-d)\ 151390000 = 378475$